Taming Large MDPs Through Stochastic Games

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November 20, 2024

Journées du GT Vérif 2024

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Motivations

- Checking multi-reachability objectives in MDPs is **computationally hard**.
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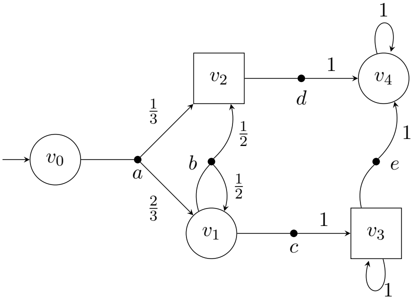
- We abstract our model as a two-player stochastic game in order to compute a lower and an upper **approximation** of the Pareto frontier.

Context



Two-player stochastic game:

- A finite set of states $V = V_1 \cup V_2$
- An initial state v_{init}
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- A probabilistic transition function $\tau : V \times A \to \mathcal{D}(V)$

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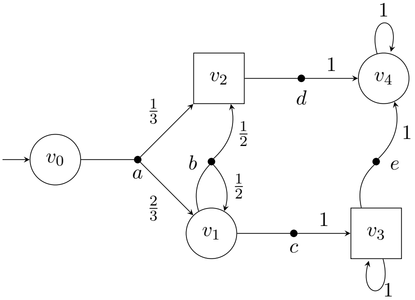

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- Plays are **infinite sequences** $\pi = v_0 a_0 v_1 a_1 \dots$ where $\tau(v_i, a_i, v_{i+1}) > 0$ for all $i \in \mathbb{N}$.
- Histories are **finite prefixes** $h = v_0 a_0 \dots a_{n-1} v_n$ of a play ending in a state, the last state of h is $\text{last}(h)$.

Strategies

A strategy for $\mathcal{P}_i$ is a function $\sigma_i : \mathsf{Hists_i}(\mathcal{G}) \to \mathcal{D}(A)$ that **respects the structure** of $\mathcal{G}$.

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We denote by $\mathcal{G}^{\sigma_1,\sigma_2}$, the Markov chain **induced** by strategies σ_1 and σ_2.
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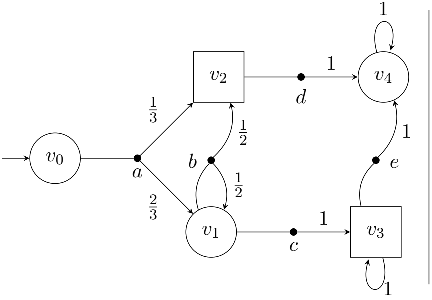

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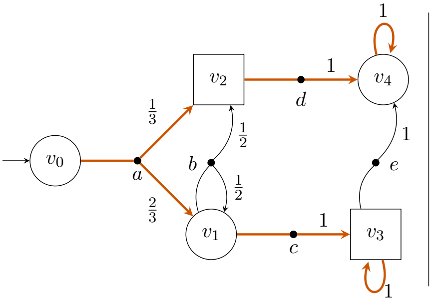

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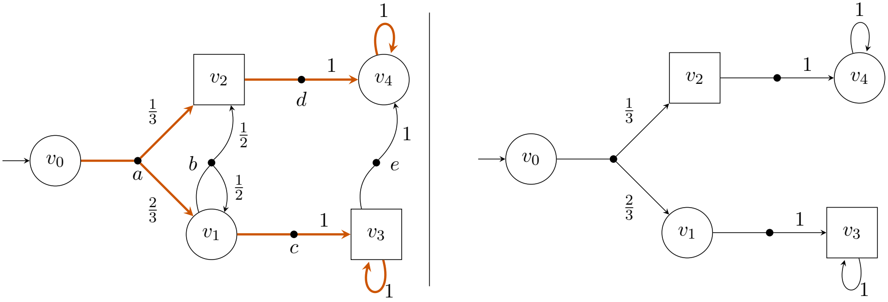

Reachability objectives

For $T \subseteq V$ a set of states, a reachability objective is defined by

$$\Diamond T = \{\pi \in \mathsf{Plays}(\mathcal{G}) \mid \exists\, i \in \mathbb{N},\, \pi[i] \in T\}.$$

- We denote by $\mathbb{P}_s^{\sigma_1,\sigma_2}(\Diamond T)$ the **probability to reach** T from s in the Markov chain induced by σ_1 and σ_2.

Multi-reachability objectives

Multi-reachability objectives are vectors of sets of states $(T_1, \ldots, T_n)$ that we want to **reach**.

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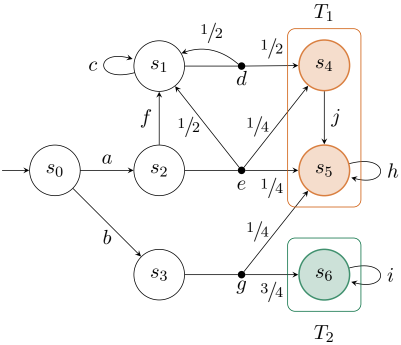

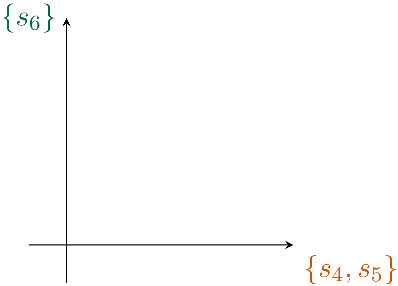

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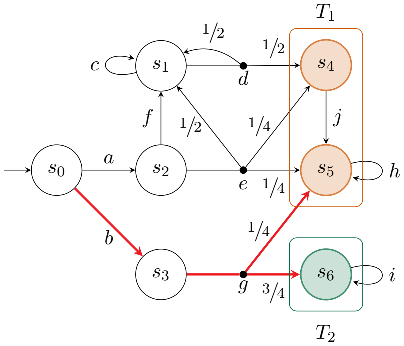

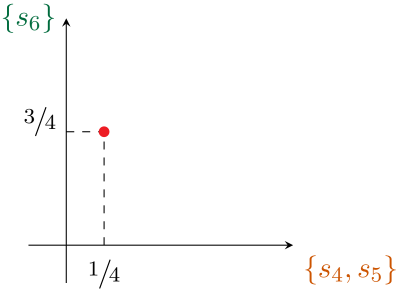

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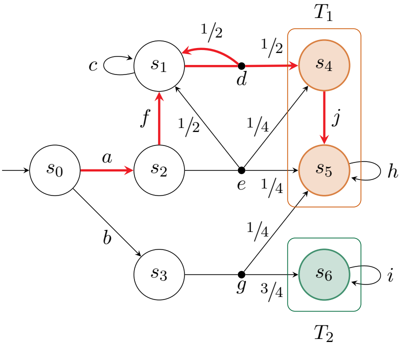

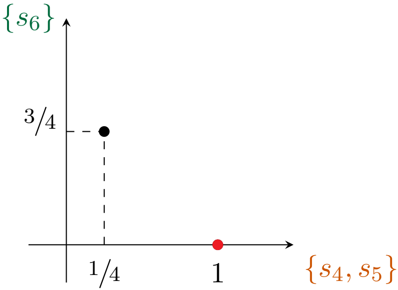

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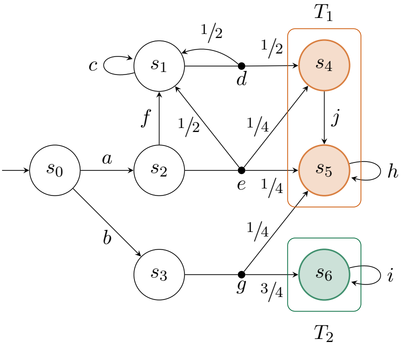

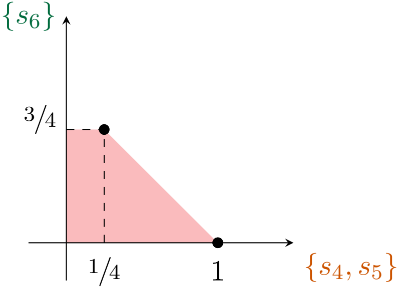

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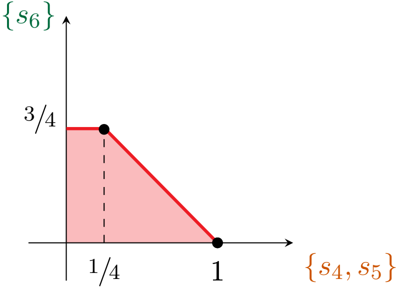
The Pareto frontier of Ach(s) is the set of points in the downward-closure of the convex hull of Ach(s) that are **not dominated**.



Game-based abstraction

We extend the work of [KKNP10][1] from one to multiple dimensions.

Goal. Abstract our MDP by **merging** states together.

$\rightsquigarrow$ Approximate the Pareto frontier through a smaller model.

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- ▷ Players are antagonistic (SG)
- ▷ Lower approximation of the Pareto frontier

Optimistic

- ▷ Players are cooperating (MDP)
- ▷ Upper approximation of the Pareto frontier

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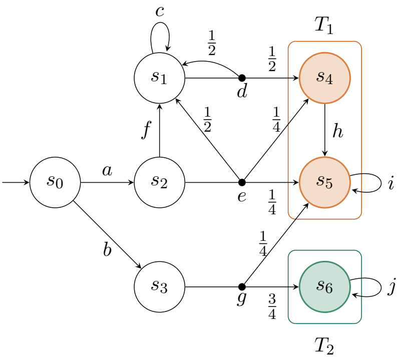
Example of GBA

Let us consider the partition $P = \{\{s_0, s_1, s_2, s_3\}, \{s_4, s_5\}, \{s_6\}\}$.

1. Lift the transition function to the partition.



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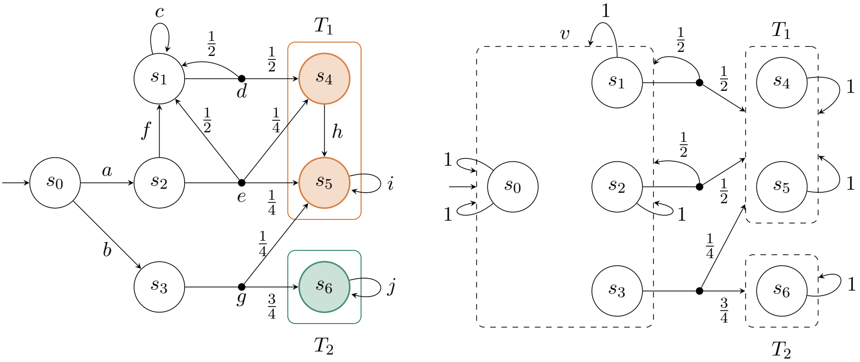

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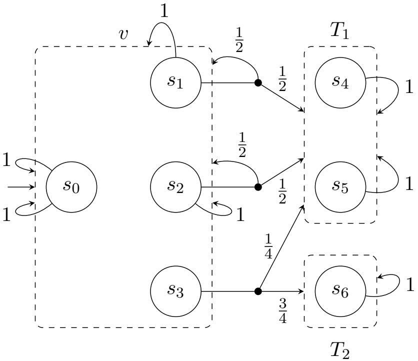

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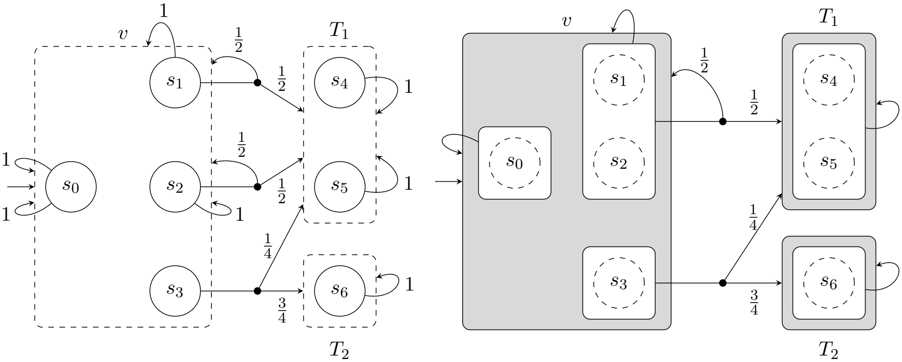
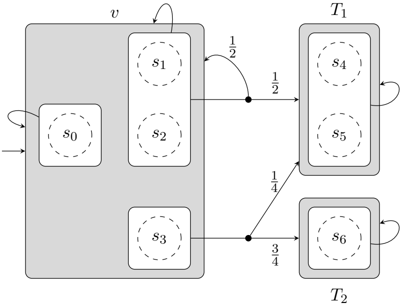


Resolving the nondeterminism

- ▷ White states are **concrete** states from the MDP,
- ▷ Grey states are **abstract** states, i.e., group of concrete states.



In a play, we alternate between:

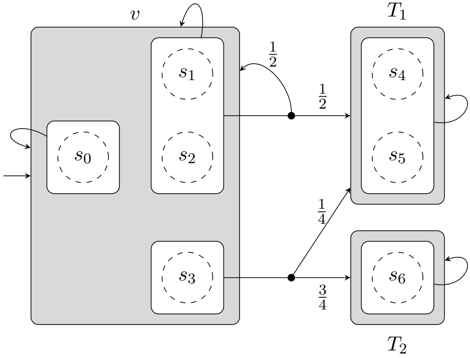
1. In an abstract state $\rightsquigarrow$ choosing a concrete state;
2. In a concrete state $\rightsquigarrow$ choosing an action of the MDP.

Resolving the nondeterminism

Depending on whether we want a lower or an upper approximation of the Pareto frontier, we have:

Optimistic: both types of states are controlled by only one player (MDP).

Pessimistic: abstract states are controlled by $\mathcal{P}_2$ and the concrete ones by $\mathcal{P}_1$ (SG).



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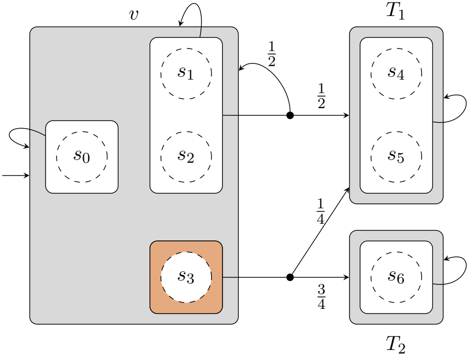
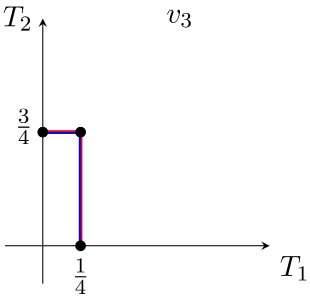
4. Repeat until the abstract states no longer need to be refined.

Computing lower and upper frontiers

- Approximating the upper frontier via [FKP12][2].
- Approximating the lower frontier using the value-iteration approach of [ACK+20][3].





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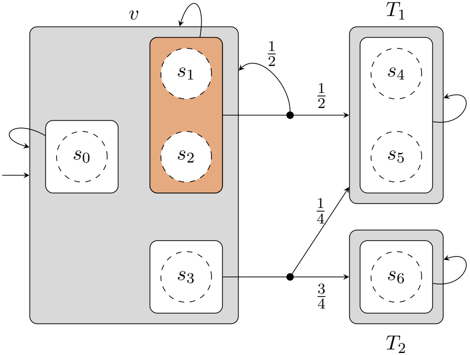

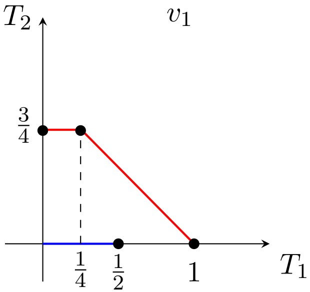

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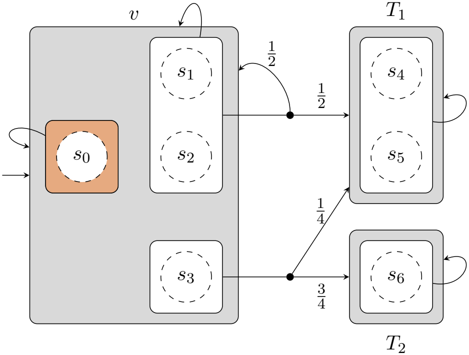

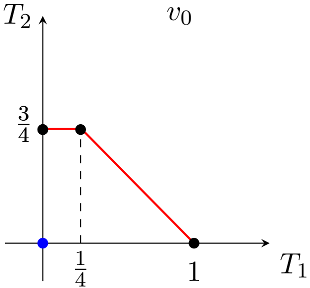

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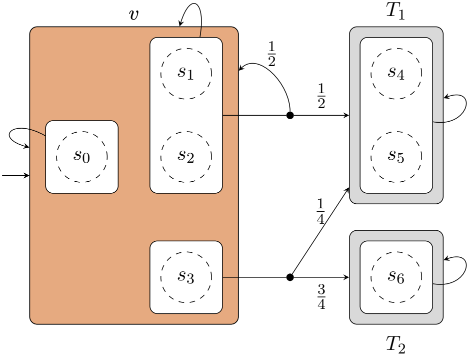

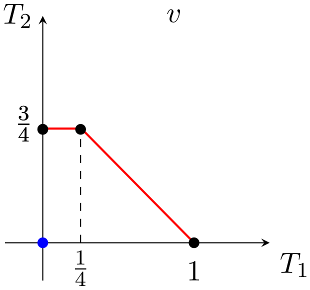

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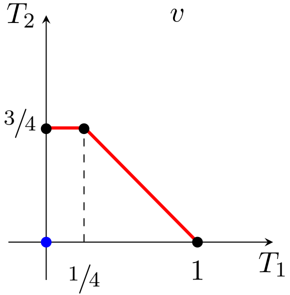

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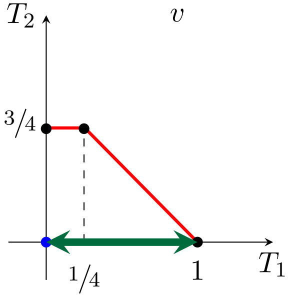

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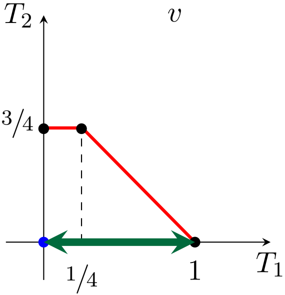

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We look at the approximations of the concrete states **contained** in v.

Splitting an abstract state

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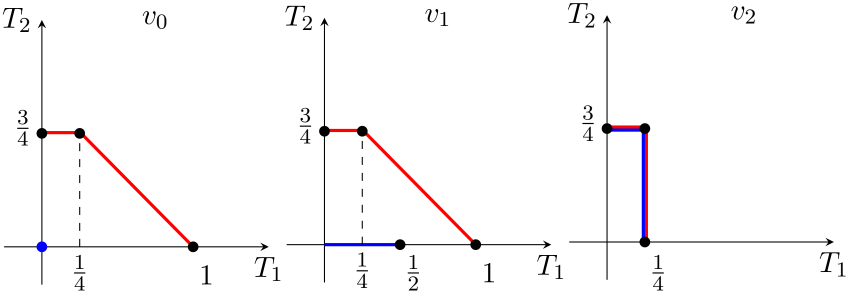

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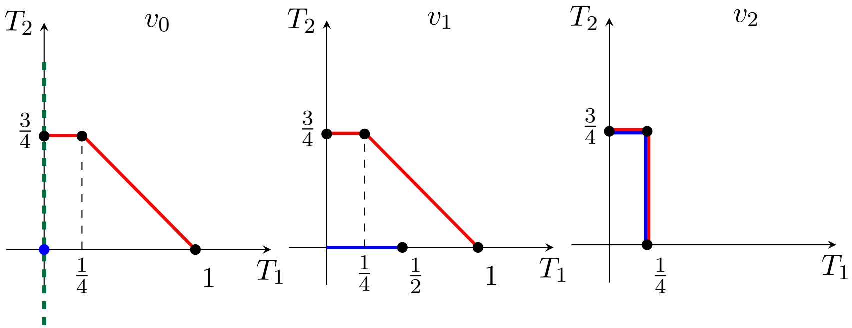

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Splitting an abstract state

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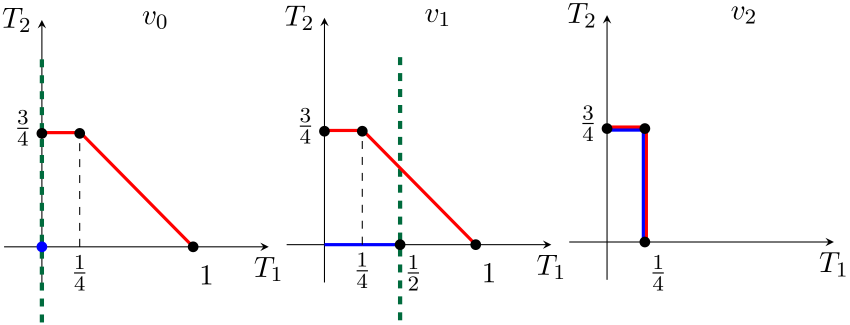

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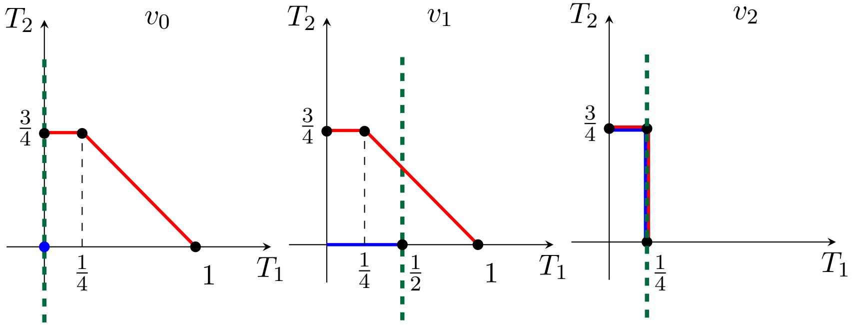

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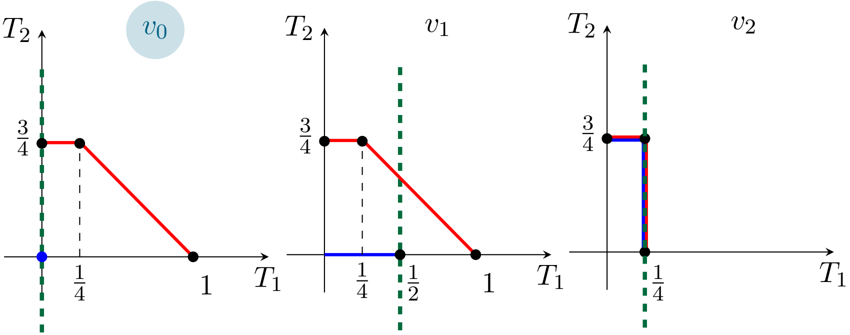

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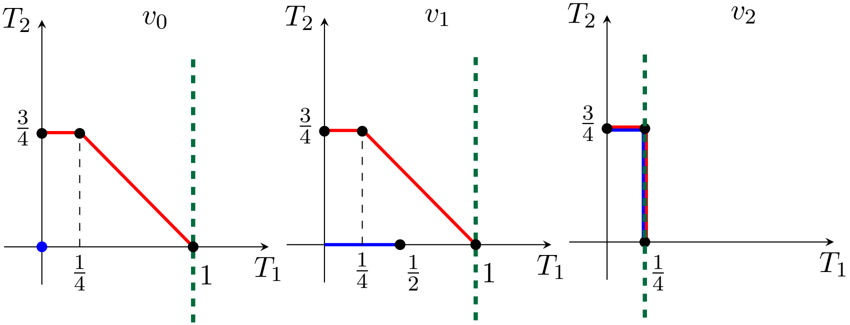

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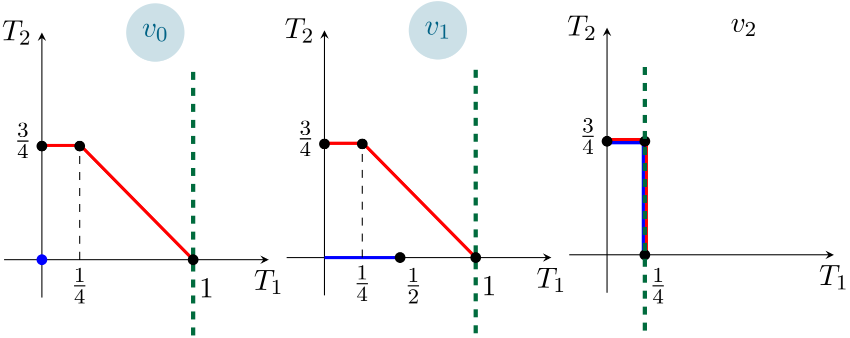
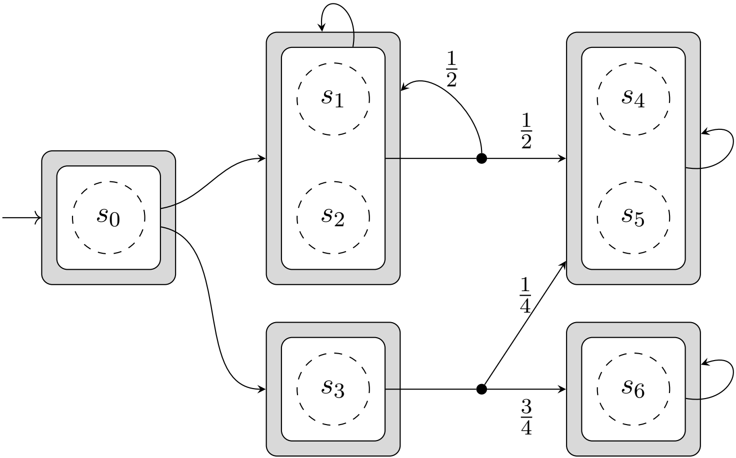


$\mathcal{S}^{\boldsymbol{d}}_{\min}(v) = \{v_0\}$ and $\mathcal{S}^{\boldsymbol{d}}_{\max}(v) = \{v_0, v_1\}$

$$\mathcal{S}^{\boldsymbol{d}}_{\min}(v) = \operatorname*{argmin}_{s \in v} \max_{\boldsymbol{x} \in s.\mathsf{L}} \langle \boldsymbol{x}, \boldsymbol{d} \rangle \quad \text{and} \quad \mathcal{S}^{\boldsymbol{d}}_{\max}(v) = \operatorname*{argmax}_{s \in v} \max_{\boldsymbol{x} \in s.\mathsf{U}} \langle \boldsymbol{x}, \boldsymbol{d} \rangle$$

New abstraction

$P_1 = \{\{s_0\}, \{s_1, s_2\}, \{s_3\}, \{s_4, s_5\}, \{s_6\}\}$



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Thank you for your attention!

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